EMBEDDING THE BICYCLIC SEMIGROUP INTO COUNTABLY COMPACT TOPOLOGICAL SEMIGROUPS

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Abstract. We study algebraic and topological properties of topological semigroups containing a copy of the bicyclic semigroup $\mathcal{C}(p,q)$. We prove that each topological semigroup S with pseudocompact square contains no dense copy of $\mathcal{C}(p,q)$. On the other hand, we construct a (consistent) example of a pseudocompact (countably compact) Tychonov semigroup containing a copy of $\mathcal{C}(p,q)$.

In this paper we study the structural properties of topological semigroups that contain a copy of the bicyclic semigroup $\mathcal{C}(p,q)$ and present a (consistent) example of Tychonov pseudocompact (countably compact) semigroup S that contains $\mathcal{C}(p,q)$. This example shows that the theorem of Koch and Wallace [18] saying that compact topological semigroups do not contain bicyclic subsemigroups cannot be generalized to the class of pseudocompact or countably compact topological semigroups. Also this example shows that the presence of an inversion is essential in a result of Gutik and Repovš [14] who proved that the bicyclic semigroup does not embed into countably compact topological inverse semigroups.

The presence or absence of a bicyclic subsemigroup in a given (topological) semigroup S has important implications for understanding the algebraic (and topological) structure of S. For example, the well-known Andersen Theorem [2], [6, 2.54] says that a simple semigroup with an idempotent but without a copy of $\mathcal{C}(p,q)$ is completely simple and hence by the Rees-Suschkewitsch Theorem [22], has the structure of a sandwich product $[X,H,Y]_\sigma$ of two sets X,Y and a group H connected by a suitable sandwich function $\sigma : Y \times X \to H$. The Rees-Suschkewitsch Theorem has also a topological version, see [4].

Having in mind the mentioned result of Koch and Wallace [18], I.I. Guran asked if the bicyclic semigroup can be embedded into a countably compact topological semigroup. In this paper we shall find many conditions on a topological semigroup S which forbid S to contain a bicyclic subsemigroup. One of the simplest conditions is the countable compactness of the square $S \times S$. On the other hand, we construct a Tychonov pseudocompact semigroup that contains a bicyclic semigroup. Moreover, assuming the existence of a countably compact abelian torsion-free topological group without convergent sequences we shall construct an example of a Tychonov countably compact topological semigroup that contains a copy of the bicyclic semigroup.

To construct such pathological semigroups, we shall study the operation of attaching a discrete semigroup D to a topological semigroup X along a homomorphism $\pi : D \to X$. This construction has two ingredients: topological and algebraic, discussed in the next four sections. In section 5 we establish some structure properties of topological semigroups that contain a copy of the bicyclic subsemigroup and in Section 6 we construct our main counterexample. Our method of constructing this counterexample is rather standard and exploits the ideas of D.Robbie, S.Svetlichny [24] (who constructed a countably compact cancelative semigroup under CH) and A.Tomita [27] (who weakened the Continuum Hypotesis in their result to a weaker version of Martin Axiom).

All topological spaces appearing in this paper are assumed to be Hausdorff.

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1. Attaching a discrete space to a topological space

In this section we describe a simple construction of attaching a discrete space D to a topological space M along a map $\pi : D \to M$ and will investigate topological properties of the obtained space $D \cup_\pi M$.

Let D be a discrete topological space and $\alpha D = D \cup \{\infty\}$ be its one-point compactification. Given a map $\pi : D \to M$ to any T_1-topological space M, consider the closed subspace

$$D \cup_\pi M = \{(x, \pi(x)) : x \in D\} \cup (\{\infty\} \times M)$$

of the product $\alpha D \times M$. We shall identify the space D with the dense discrete subspace $\{(x, \pi(x)) : x \in D\}$ and M with the closed subspace $\{\infty\} \times M$ of $D \cup_\pi M$. Let $\bar{\pi} = \pi \cup \mathrm{id}_M : D \cup_\pi M \to M$ denote the projection to the second factor. Observe that the topology of the space $D \cup_\pi M$ is the weakest topology that induces the original topologies on the subspaces D and M of $D \cup_\pi M$ and makes the map $\bar{\pi}$ continuous.

The following (almost trivial) propositions describe some elementary properties of the space $D \cup_\pi M$.

Proposition 1.1. *If the space M is Hausdorff (or Tychonov), then so is the space $D \cup_\pi M$.*

Proposition 1.2. *If M is (separable) metrizable and D is countable, then the space $D \cup_\pi M$ is (separable) metrizable too.*

Proposition 1.3. *If the space M is compact, then so is the space $D \cup_\pi M$.*

We recall that a topological space X is *countably compact* if each countable open cover of X has a finite subcover. This is equivalent to saying that the space X contains no infinite closed discrete subspace.

Proposition 1.4. *If some power M^κ of the space M is countably compact, then the power $(D \cup_\pi M)^\kappa$ is countably compact too.*

Proof. Since $D \cup_\pi M$ is a closed subspace of $\alpha D \times M$, the power $(D \cup_\pi M)^\kappa$ is a closed subspace of $(\alpha D \times M)^\kappa$. So, it suffices to check that the latter space is countably compact. Since the product of a countably compact space and a compact space is countably compact [10, 3.10.14], the product $M^\kappa \times (\alpha D)^\kappa$ is compact and so is its topological copy $(\alpha D \times M)^\kappa$. □

If the space M is Tychonov, then $D \cup_\pi M$ is a subspace of the compact Hausdorff space $D \cup_\pi \beta M$ where βM is the Stone-Čech compactification of M. Assuming that M is countably compact at $\pi(D)$ we shall show that $D \cup_\pi \beta M$ coincides with the Stone-Čech compactification of $D \cup_\pi M$.

We shall say that a topological space X is *countably compact* at a subset $A \subset X$ if each infinite subset $B \subset A$ has an accumulation point x in X. The latter means that each neighborhood $O(x)$ of x contains infinitely many points of the set B.

Proposition 1.5. *If the space M is Tychonov and is countably compact at the subset $\pi(D)$, then $D \cup_\pi \beta M$ is a Stone-Čech compactification of $D \cup_\pi M$.*

Proof. The space $D \cup_\pi M$ is Tychonov, being a subspace of the Tychonov space $\alpha D \times M$ and hence has a Stone-Čech compactification $\beta(D \cup_\pi M)$. Since the space M is a retract of $D \cup_\pi M$, the Stone-Čech compactification βM is a retract of $\beta(D \cup_\pi M)$. Let $\beta i : \beta(D \cup_\pi M) \to D \cup_\pi \beta(M)$ be the Stone-Čech extension of the identity inclusion $i : D \cup_\pi M \to D \cup_\pi \beta(M)$. We claim that βi is a homeomorphism.

First we show that the subset $D \cup \beta M \subset \beta(D \cup_\pi M)$ is compact. Indeed, given an open cover $\mathcal{U}$ of $D \cup \beta M$ we can find a finite subcover $\mathcal{V} \subset \mathcal{U}$ of βM and then consider the set $D' = D \setminus \cup \mathcal{V}$. We claim that this set D' is finite. Assuming the converse and using the countable compactness of M at $\pi(D)$ we could find a point $a \in M$ such that for every neighborhood $O(a) \subset M$ the set $\{x \in D' : \pi(x) \in O(a)\}$ is infinite. Take any open set $V \in \mathcal{V}$ containing the point a. By the definition of the topology on $D \cup_\pi M$ there is a neighborhood $O(a) \subset M \cap V$ of a and a finite subset $F \subset D$ such that $\bar{\pi}^{-1}(O(a)) \setminus F \subset V$.

Then the set $\{x \in D' : \pi(x) \in O(a)\}$ lies in F and hence is finite, which is a contradiction. Hence the set D' is finite and we can find a finite subfamily $\mathcal{W} \subset \mathcal{U}$ with $D' \subset \cup \mathcal{W}$. Then $\mathcal{V} \cup \mathcal{W} \subset \mathcal{U}$

is a finite subcover of $D \cup \beta M$. Now we see that the subset $D \cup \beta M$, being compact and dense in $\beta(D \cup_\pi M)$, coincides with $\beta(D \cup_\pi M)$. It follows that the continuous map $\beta i = \beta i | D \cup \beta M$ is bijective and hence is a homeomorphism. □

Following A.V. Arkhangel'skii [1, III.§4], we define a topological space X to be *countably pracompact* if X is countably compact at a dense subset of X. It is clear that each countably compact space is countably pracompact.

Proposition 1.6. *The space $D \cup_\pi M$ is countably pracompact if and only if M is countably compact at a dense subset $A \supset \pi(D)$ of M.*

Proof. If the space $D \cup_\pi M$ is countably pracompact, then it is countably compact at some dense subset $A \subset D \cup_\pi M$. The set A, being dense, contains the open discrete suspace D of $D \cup_\pi M$. The continuity of the retraction $\bar{\pi} : D \cup_\pi M \to M$ implies that the space M is countably compact at the dense subset $\bar{\pi}(A) \supset \pi(D)$ of M, so M is countably pracompact.

Now assume conversely that the space M is countably compact at a dense subset $A \supset \pi(D)$. We claim that $D \cup_\pi M$ is countably compact at the dense subset $D \cup A$. We need to check that each infinite subset $B \subset D \cup A$ has a cluster point in $D \cup_\pi M$. If $B \cap A$ is infinite, then the set $B \cap A \subset B$ has an accumulation point in M because M is countably compact at A. If $B \cap A$ is finite, then $B \setminus A$ is infinite and then the projection $\pi(B \setminus A)$ has an accumulation point x at M because of the countable compactness of M at $\pi(D) \subset A$. By the definition of the topology on $D \cup_\pi M$, the point x is an accumulation point of the set $B \setminus A$. □

A topological space X is defined to be *pseudocompact* if each locally finite open cover of X is finite. According to [10, 3.10.22] a Tychonov space X is pseudocompact if and only if each continuous real-valued function on X is bounded. For each topological space we have the following implications

$$\text{countably compact} \Rightarrow \text{countably pracompact} \Rightarrow \text{pseudocompact}.$$

Proposition 1.7. *The space $D \cup_\pi M$ is pseudocompact if and only if M is pseudocompact and M countably compact at the subset $\pi(D) \subset M$.*

Proof. Assume that the space $D \cup_\pi M$ is pseudocompact. To prove that M is pseudocompact, take any locally finite open cover $\mathcal{U}$ of M. By the continuity of the map $\bar{\pi} = \mathrm{id} \cup \pi : D \cup_\pi M \to M$, the family $\mathcal{V} = \{\bar{\pi}^{-1}(U) : U \in \mathcal{U}\}$ is a locally finite open cover of $D \cup_\pi M$. The pseudocompactness of $D \cup_\pi M$ guarantees that the family $\mathcal{V}$ is finite and so is the family $\mathcal{U}$.

Next, we prove that M is countably compact at $\pi(D)$. Assuming the converse, we could find a sequence $D' = \{x_n : n \in \omega\} \subset D$ such that $\pi(x_n) \neq \pi(x_m)$ for $n \neq m$ and the image $\pi(D')$ is closed and discrete in M. Define an unbounded function $f : D \cup_\pi M \to \mathbb{R}$ letting

$$f(x) = \begin{cases} n & \text{if } x = x_n \text{ for some } n \in \omega \\ 0 & \text{otherwise,} \end{cases}$$

and check that f is continuous, which contradicts the pseudocompactness of $D \cup_\pi M$.

To prove the "if" part, assume that the space M is pseudocompact and is countably compact at the subset $\pi(D)$. To prove that the space $D \cup_\pi M$ is pseudocompact, fix a locally finite open cover $\mathcal{U}$ of $D \cup_\pi M$ and consider the locally finite open subcover $\mathcal{V} = \{U \in \mathcal{U} : U \cap M \neq \emptyset\}$ of M. The pseudocompactness of M guarantees that the cover $\mathcal{V}$ is finite. Repeating the argument of the proof of Proposition 1.5, we can check that the set $D' = D \setminus \cup\mathcal{V}$ is finite. The local finity of the family $\mathcal{U}$ implies that the family $\mathcal{W} = \{U \in \mathcal{U} : U \cap D' \neq \emptyset\}$ is finite. Since $\mathcal{U} = \mathcal{V} \cup \mathcal{W}$, the cover $\mathcal{U}$ of $D \cup_\pi M$ is finite. □

Following [3], we define a topological space X to be *openly factorizable* if every continuous map $f : X \to Y$ to a metrizable separable space Y can be written as the composition $g \circ \pi$ of an open continuous map $p : X \to K$ onto a metrizable separable space K and a continuous map $g : K \to Y$.

Proposition 1.8. *If the set D is countable and M is openly factorizable, then the space $D \cup_\pi M$ is openly factorizable too.*

Proof. Fix any continuous map $f : D \cup_\pi M \to Y$ to a metrizable separable space Y. Since M is openly factorizable, there are an open continuous map $p : M \to K$ onto a separable metrizable space K and a continuous map $g : K \to Y$ such that $f|M = g \circ p$.

Consider the map $p\pi = p \circ \pi : D \to K$ and the corresponding space $D \cup_{p\pi} K$ that is separable and metrizable by Proposition 1.2. Let $\bar{p} = \mathrm{id} \cup p : D \cup \pi M \to D \cup_{p\pi} K$ be the map that is identity on D and coincides with the map p on M. It follows from the openness of the map p that the map $\bar{p}$ is open.

Now extend the map $g : K \to Y$ to a map $\bar{g} : D \cup_{p\pi} K \to Y$ letting $\bar{g}|D = f|D$. It is easy to see that $f = \bar{g} \circ \bar{p}$. It remains to check that the map $\bar{g}$ is continuous. Take any open set $U \subset Y$ and observe that $\bar{g}^{-1}(U) = \bar{p}(f^{-1}(U))$ because f is continuous and $\bar{p}$ is open. □

2. Compact extensions of topological semigroups

In this section we survey some known results on compact extensions of semitopological semigroups. By a *semitopological semigroup* we understand a topological space S endowed with a separately continuous semigroup operation $* : S \times S \to S$. If the operation is jointly continuous, then S is called a *topological semigroup*.

Let $\mathcal{C}$ be a class of compact Hausdorff semitopological semigroups. By a *$\mathcal{C}$-compactification* of a semitopological semigroup S we understand a pair $(\mathcal{C}(S), \eta)$ consisting of a compact semitopological semigroup $\mathcal{C}(S) \in \mathcal{C}$ and a continuous homomorphism $\eta : S \to \mathcal{C}(S)$ (called the *canonic homomorphism*) such that for each continuous homomorphism $h : S \to K$ to a semitopological semigroup $K \in \mathcal{C}$ there is a unique continuous homomorphism $\bar{h} : \mathcal{C}(S) \to K$ such that $h = \bar{h} \circ \eta$. It follows that any two $\mathcal{C}$-compactifications of S are topologically isomorphic.

We shall be interested in $\mathcal{C}$-compactifications for the following classes of semigroups:

- WAP of compact semitopological semigroups;
- AP of compact topological semigroups;
- SAP of compact topological groups.

The corresponding $\mathcal{C}$-compactifications of a semitopological semigroup S will be denoted by $\mathrm{WAP}(S)$, $\mathrm{AP}(S)$, and $\mathrm{SAP}(S)$. The notation came from the abbreviations for weakly almost periodic, almost periodic, and strongly almost periodic function rings that determine those compactifications, see [5, Ch.IV], [25, Ch.III], [17, §21].

The inclusions SAP ⊂ AP ⊂ WAP induce canonic homomorphisms

$$\eta : S \to \mathrm{WAP}(S) \to \mathrm{AP}(S) \to \mathrm{SAP}(S)$$

for any semitopological semigroup S. It should be mentioned that the canonic homomorphism $\eta : S \to \mathrm{WAP}(S)$ needs not be injective. For example, for the group $H_+[0,1]$ of orientation-preserving homeomorphisms of the interval the WAP-compactification is a singleton, see [21]. However, for countably compact semitopological semigroups the situation is more optimistic. The following two results are due to E.Reznichenko [23].

Theorem 2.1 (Reznichenko)**.** *For any Tychonov countably compact semitopological semigroup S the semigroup operation of S extends to a separately continuous semigroup operation on βS, which implies that βS coincides with the WAP-compactification of S.*

The same conclusion holds for Tychonov pseudocompact topological semigroups.

Theorem 2.2 (Reznichenko)**.** *For any Tychonov pseudocompact topological semigroup S the semigroup operation of S extends to a separately continuous semigroup operation βS, which implies that βS coincides with the WAP-compactification of S.*

This theorem combined with the Glicksberg Theorem [10, 3.12.20(c)] on the Stone-Čech compactifications of products of pseudocompact spaces, implies the following important result, see [3, 1.3].

Theorem 2.3. *For any Tychonov topological semigroup S with pseudocompact square $S \times S$ the semigroup operation of S extends to a continuous semigroup operation βS, which implies that βS coincides with the AP-compactification of S.*

Another result of the same spirit involves openly factorizable spaces with weakly Lindelöf squares. We recall that a topological space X is *weakly Lindelöf* if each open cover $\mathcal{U}$ of X contains a countable subcollection $\mathcal{V} \subset \mathcal{U}$ whose union $\cup\mathcal{V}$ is dense in X. The following extension theorem is proved in [3].

Theorem 2.4. *For any Tychonov openly factorizable topological semigroup S with weakly Lindelöf square $S \times S$ the semigroup operation of S extends to a continuous semigroup operation βS, which implies that βS is an AP-compactification of S.*

The following theorem also is proved in [3]. It gives conditions on a pseudocompact topological semigroup S guaranteeing that its Stone-Čech compactification βS coincides with the SAP-compactification $\mathrm{SAP}(S)$ of S.

Theorem 2.5. *The Stone-Čech compactification βS of a Tychonov pseudocompact topological semigroup S is a compact topological group if one of the following conditions holds:*

1. *S contains a totally bounded topological group as a dense subgroup;*
2. *S contains a dense subgroup and $S \times S$ is pseudocompact.*

3. Attaching a discrete semigroup to a semitopological semigroup

In this section we extend the construction of the space $D \cup_\pi M$ to the category of semitopological semigroups and their continuous homomorphisms.

Given a homomorphism $\pi : D \to M$ from a discrete semigroup D into a semitopological semigroup M extend the semigroup operations from $(D, \cdot)$ and $(M, \cdot)$ to $D \cup_\pi M$ letting

$$xy = \begin{cases} x \cdot y & \text{if } x, y \in D \text{ or } x, y \in M, \\ \pi(x) \cdot y & \text{if } x \in D \text{ and } y \in M, \\ x \cdot \pi(y) & \text{if } x \in M,\ y \in D. \end{cases}$$

Endowed with the so-extended operation, the space $S = D \cup_\pi M$ becomes a semitopological semigroup containing D and M as subsemigroups. Moreover, the map $\bar\pi = \pi \cup \mathrm{id}_M : D \cup_\pi M \to M$ is a continuous semigroup homomorphism.

Now we will find some conditions guaranteeing that $S = D \cup_\pi M$ is a topological semigroup.

Definition 3.1. A homomorphism $\pi : D \to M$ is called *finitely resolvable* if for every $x, y \in M$ and $z \in D$ the set $\{(u, v) \in D \times D : \pi(u) = x,\ \pi(v) = y,\ xy = z\}$ is finite.

Observe that each one-to-one homomorphism is finitely resolvable.

Theorem 3.2. *Let $\pi : D \to M$ be a homomorphism from a discrete semigroup to a topological semigroup M. The semitopological semigroup $S = D \cup_\pi M$ is a topological semigroup provided the following conditions all are satisfied:*

1. *the homomorphism π is finitely resolvable;*
2. *the image $\pi(D)$ is discrete in M;*
3. *the complement $M \setminus \pi(D)$ is a two-sided ideal in M.*

Proof. To prove the continuity of the semigroup operation on $S = D \cup_\pi M$, fix any two points $x, y \in S$. If x or y belongs to the discrete subspace D of S, then the semigroup operation is continuous at (x, y) because S is a semitopological semigroup. So, we assume that $x, y \in M$. Two cases are possible.

1. The points x, y belong to $\pi(D)$ and then $xy \in \pi(D)$ because π is a homomorphism. Let $O(z)$ be any neighborhood of z in S. By the definition of the topology on $S = D \cup_\pi M$, there is an open neighborhood $U(z) \subset M$ and a finite subset $F \subset D$ such that $\bar\pi^{-1}(U(z)) \setminus F \subset O(z)$. Since the homomorphism π is finitely resolvable, the set $A = \{(u, v) \in D \times D : \pi(u) = x,\ \pi(v) = y,\ uv \in F\}$ is finite. Find a finite subset $B \subset D$ such that $A \subset B \times B$.

By the continuity of the semigroup operation on M, we can find open neighborhoods $U(x), U(y) \subset M$ of the points x, y such that $U(x) \cdot U(y) \subset U(z)$. Moreover, since $\pi(D)$ is discrete, those neighborhoods can be chosen so that $U(x) \cap \pi(D) = \{x\}$ and $U(y) \cap \pi(D) = \{y\}$. Now consider the open neighborhoods $O(x) = \bar{\pi}^{-1}(U(x)) \setminus B$ and $O(y) = \bar{\pi}^{-1}(U(y)) \setminus B$ of the points x, y in S. We claim that $O(x) \cdot O(y) \subset O(z)$.

Indeed, take any points $u \in O(x)$, $v \in O(y)$. If $u, v \in M$, then $u \in U(x)$, $v \in U(y)$ and hence $uv \in U(x) \cdot U(y) \subset U(z) \subset O(z)$ by the choice of the neighborhoods $U(x), U(y)$ and $U(z)$. If $u \in M$ and $v \in D$, then $u \in U(x)$, $\pi(v) \in U(y)$ and then $uv = u\pi(v) \in U(x) \cdot U(y) \subset U(z) \subset O(z)$. The same argument works if $u \in D$ and $v \in M$.

Finally, assume that $u, v \in D$. Then $\pi(u) \in U(x) \cap \pi(D) = \{x\}$. By the same reason, $\pi(v) = y$. Consequently, $\pi(uv) = xy = z$. Since $(u, v) \notin B \times B \supset A$, the product $uv \notin F$. Consequently, $uv \in \bar{\pi}^{-1}(z) \setminus F \subset O(z)$.

2. Next we consider the second case: one of the points x or y does not belong to $\pi(D)$. Since $M \setminus \pi(D)$ is a two-sided ideal in M, we get $z = xy \in M \setminus \pi(D)$. Let $O(z) \subset S$ be any neighborhood of z. By the definition of the topology on S, we can find a neighborhood $U(z) \subset M$ of z and a finite subset $F \subset D$ such that $\bar{\pi}^{-1}(U(z)) \setminus F \subset O(z)$. Since $z \notin \pi(F)$, we can assume that $U(z) \cap \pi(F) = \emptyset$ and hence $\bar{\pi}^{-1}(U(z)) \subset O(z)$. By the continuity of the semigroup operation of M there are neighborhoods $U(x), U(y) \subset M$ of the points x, y such that $U(x) \cdot U(y) \subset U(z)$. Then $O(x) = \bar{\pi}^{-1}(U(x))$ and $O(y) = \bar{\pi}^{-1}(U(y))$ are two neighborhoods of the points x, y in S such that $O(x) \cdot O(y) \subset \bar{\pi}^{-1}(U(z)) \subset O(z)$, which completes the proof. □

4. Attaching the bicyclic semigroup to a topological semigroup

In this section we study the structure of the semigroups $D \cup_\pi M$ in the case $D = \mathcal{C}(p, q)$ is the bicyclic semigroup. The bicyclic group plays an important role in the structure theory of semigroups, see [6]. A remarkable property of this semigroup is that it is non-topologizable in the sense that any Hausdorff topology turning $\mathcal{C}(p, q)$ into a topological semigroup is discrete [9].

The bicyclic semigroup $\mathcal{C}(p, q)$ is generated by two element p, q and one relation $qp = 1$, see [6]. It follows that each element of $\mathcal{C}(p, q)$ can be uniquely written as the product $p^n q^m$ for some $n, m \in \omega$. The element $1 = p^0 q^0$ is a two-sided unit for $\mathcal{C}(p, q)$. The product $p^m q^n \cdot p^i q^j$ of two elements of the bicyclic semigroup $\mathcal{C}(p, q)$ is equal to $p^m q^{n-i+j}$ if $n \geq i$ and to $p^{m+i-n} q^j$ if $n \leq i$. The semigroup $\{p^n q^n : n \in \omega\}$ of the idempotents of $\mathcal{C}(p, q)$ is isomorphic to the semigroup ω of finite ordinals endowed with the operation of maximum.

If $\pi : \mathcal{C}(p, q) \to H$ is any homomorphism of $\mathcal{C}(p, q)$ into a group, then $\pi(1)$ is an idempotent e of the group H and the relation $qp = 1$ implies that $\pi(q)$ and $\pi(p)$ are mutually inverse elements of H, generating a cyclic subgroup of H, see [6, 1.32].

Now we are able to present the main result of this section.

Theorem 4.1. *Let $\pi : \mathcal{C}(p, q) \to M$ be a homomorphism of the bicyclic semigroup into a topological semigroup M such that $Z = \pi(\mathcal{C}(p, q))$ is an infinite cyclic subgroup of M and M is countably compact at Z. The semitopological semigroup $S = \mathcal{C}(p, q) \cup_\pi M$ has the following properties:*

1. *If M is Tychonov, compact, countably compact, countably pracompact or pseudocompact, then so is the space S;*
2. *S is a topological semigroup if and only if for every $c \in \mathcal{C}(p, q)$ the set $A_c = \{(x, y) \in \mathcal{C}(p, q)^2 : xy = c\}$ is closed and discrete in $S \times S$;*
3. *S is a topological semigroup provided the subgroup Z is discrete in M and $M \setminus Z$ is an ideal in M;*
4. *If S is a topological semigroup, then the square $S \times S$ is not pseudocompact.*

Proof. 1. The first item follows from Propositions 1.1—1.7.

2. Assume that S is a topological semigroup and take any $c \in \mathcal{C}(p, q)$. We need to check that the set $A_c = \{(x, y) \in \mathcal{C}(p, q)^2 : xy = c\}$ is closed and discrete in $S \times S$. Assuming the converse, we

could find an accumulating point $(a,b) \in S\times S$ for the set A. It follows that $a,b\in M$ and hence $ab\in M\subset S\setminus\{c\}$. By the continuity of the semigroup operation on S, the points a,b have open neighborhoods $O(a),O(b)\subset S$ such that $O(a)\cdot O(b)\subset S\setminus\{c\}$. Since the pair (a,b) is an accumulation point for the set A, there is a pair $(x,y)\in A\cap(O(a)\times O(b))$. It follows that $c=xy\in O(a)\cdot O(b)\subset S\setminus\{c\}$ which is a contradiction.

Now we shall show that S is a topological semigroup under the assumption that for every $c\in\mathcal{C}(p,q)$ the set A_c is closed and discrete in $S\times S$. We need to check the continuity of the multiplication at each pair $(x,y)\in S\times S$. If x or y belongs to D, then this follows from the continuity of left and right shifts on S. So, we can assume that $x,y\in M$. Let $z=xy$ and $O(z)\subset S$ be an open neighborhood of z. It follows from the definition of the topology of $S=D\cup_\pi M$ that there are a neighborhood $U(z)\subset M$ and a finite subset $F\subset\mathcal{C}(p,q)$ such that $\bar\pi^{-1}(U(z))\setminus F\subset O(z)$. By the continuity of the semigroup operation on M the points x,y have neighborhoods $U(x),U(y)\subset M$ such that $U(x)\cdot U(y)\subset U(z)$.

Since the set $A=\bigcup_{c\in F}A_c$ is closed and discrete in $S\times S$ and $(x,y)\notin A$ (because $(x,y)\in M\times M$), we can find neighborhoods $O(x)\subset\bar\pi^{-1}(U(x))$ and $O(y)\subset\bar\pi^{-1}(U(y))$ of the points x,y such that the set $O(x)\times O(y)$ is disjoint with the set A. In this case $O(x)\cdot O(y)\subset S\setminus F$ and $O(x)\cdot O(y)\subset \bar\pi^{-1}(U(x))\cdot\bar\pi^{-1}(U(y))\subset\bar\pi^{-1}(U(z))$, which implies $O(x)\cdot O(y)\subset\bar\pi^{-1}(U(z))\setminus F\subset O(z)$.

3. Assume that the subspace Z of M is discrete and $M\setminus Z$ is a two-sided ideal of M. Theorem 3.2 will imply that S is a topological semigroup as soon as we check that the homomorphism $\pi:\mathcal{C}(p,q)\to Z$ is finitely resolvable. The verification of the finite resolvability of π is elementary and is left as an exercise to the reader.

4. Assume that S is a topological semigroup. By the item (2), for every $c\in\mathcal{C}(p,q)$ the set $D_c=\{(x,y)\in\mathcal{C}(p,q)^2: xy=c\}$ is closed and discrete in $S\times S$. Since $\mathcal{C}(p,q)$ is an open subset of S, the set D_c is closed-and-open in $S\times S$. Then we can take any unbounded function $f:D_c\to[0,\infty)$ and extend it to $S\times S$ letting $f(x,y)=0$ if $(x,y)\notin D_c$. In such a way we construct a continuous unbounded function witnessing that $S\times S$ is not pseudocompact. □

5. The structure of topological semigroups that contain bicyclic semigroups

In fact, many properties of the topological semigroups $\mathcal{C}(p,q)\cup_\pi M$, established in Theorem 4.1 hold for any topological semigroup containing a (dense) copy of the bicyclic semigroup $\mathcal{C}(p,q)$.

Theorem 5.1. *If a topological semigroup S contains the bicyclic semigroup $\mathcal{C}(p,q)$ as a dense subsemigroup, then*

(1) *the complement $S\setminus\mathcal{C}(p,q)$ is a two-sided ideal in S;*
(2) *for every $c\in\mathcal{C}(p,q)$ the set $D_c=\{(x,y): x,y\in\mathcal{C}(p,q),\ xy=c\}$ is a closed-and-open discrete subspace of $S\times S$;*
(3) *the square $S\times S$ is not pseudocompact;*
(4) *βS is not openly factorizable;*
(5) *the almost periodic compactification $\mathrm{AP}(S)$ of S is a compact topological group and hence the canonic homomorphism $\eta:S\to\mathrm{AP}(S)$ is not injective.*

Proof. 1. The fact that $S\setminus\mathcal{C}(p,q)$ is a two-sided ideal in S wasproved by Eberhard and Selden in [9].

2. Given any point $c\in\mathcal{C}(p,q)$ we should check that the set $D_c=\{(x,y)\in\mathcal{C}(p,q)^2: xy=c\}$ is an open-and-closed discrete subspace of $S\times S$. By [9], the topology on $\mathcal{C}(p,q)$ induced from S is discrete. Consequently, the subspace $\mathcal{C}(p,q)$, being discrete and dense in S, is open S. Then the square $\mathcal{C}(p,q)\times\mathcal{C}(p,q)$ is open and discrete in $S\times S$ and so is its subspace D_c. It remains to check that the set D_c is closed in $S\times S$. Assuming the opposite, find an accumulation point $(a,b)\in S\times S$ of the subset D_c. The continuity of the semigroup operation implies that $ab=c$. On the other hand, since the space $\mathcal{C}(p,q)\times\mathcal{C}(p,q)$ is discrete, the points a,b belong to the ideal $S\setminus\mathcal{C}(p,q)$ and hence $ab\in S\setminus\mathcal{C}(p,q)$ cannot be equal to c.

3. The space $S\times S$ fails to be pseudocompact because it contains an infinite closed-and-open discrete subspace D_c, $c\in\mathcal{C}(p,q)$.

4. Assuming that the Stone-Čech compactification βS of S is openly factorizable, we may apply Proposition 2.3 of [3] to conclude that S is an openly factorizable pseudocompact space. Since the space S has separable and hence has weakly Lindelöf square, we can apply Theorem 2.4 to conclude that βS is a compact topological semigroup that contains the bicyclic semigroup. But this is forbidden by the Hildenbrandt-Koch Theorem [16].

5. Let $\eta : S \to \mathrm{AP}(S)$ be the homomorphism of S into its almost periodic compactification. The restriction $\eta|\mathcal{C}(p,q)$ cannot be injective because compact topological semigroups do not contain bicyclic semigroups. Consequently, the image $Z = \eta(\mathcal{C}(p,q))$ is a cyclic subgroup of $\mathrm{AP}(S)$ by Corollary 1.32 of [6]. Since $\mathcal{C}(p,q)$ is dense in S, the subgroup Z is dense in $\mathrm{AP}(S)$. Now Theorem 2.5(2) guarantees that $\mathrm{AP}(S)$ is a compact topological group. □

The following theorem extends (and corrects) Theorem 2.6 of [13].

Theorem 5.2. *Let S be a topological semigroup containing the bicyclic semigroup $\mathcal{C}(p,q)$ as a dense subsemigroup. If the space S is countably compact at the set $E_C = \{p^nq^n : n \in \omega\}$ of the idempotents of $\mathcal{C}(p,q)$, then*

1. *the closure $\bar{E}_C$ of the set E_C in S is compact and has a unique non-isolated point e that commutes with all elements of S;*
2. *the map $\pi : S \to S$, $\pi : x \mapsto x \cdot e = e \cdot x$, is a continuous homomorphism that retracts S onto the ideal $M = S \setminus \mathcal{C}(p,q)$ having the idempotent e as a two-sided unit;*
3. *the element $a = \pi(p)$ generates a dense cyclic subgroup Z of M;*
4. *$\pi(p^nq^m) = a^{n-m}$ for all $n, m \in \omega$;*
5. *$\lim_{n\to\infty} p^{n+k}q^n = a^k$ for every $k \in \mathbb{Z}$.*
6. *If S is countably compact at $\mathcal{C}(p,q)$, then S is countably pracompact and the cyclic subgroup Z contains no non-trivial sequence that converges in S;*
7. *If S is countably compact at $\mathcal{C}(p,q)$ and M is Tychonov, then*
   - (a) *S is topologically isomorphic to $\mathcal{C}(p,q) \cup_\pi M$;*
   - (b) *M is not openly factorizable;*
   - (c) *$M \times M$ is not pseudocompact;*
   - (d) *M contains no dense totally bounded topological subgroup.*

Proof. 1. The set $E_C = \{p^nq^n : n \in \omega\}$ of the idempotents of the bicyclic semigroup $\mathcal{C}(p,q)$ has an accumulation point $e \in \bar{E}_C$ because S is countably compact at E_C. We claim that this accumulation point e is unique. Assume conversely that E_C has another accumulation point $e' \neq e$. Then the product ee' differs from e or e'. We lose no generality assuming that $ee' \neq e'$. Since S is Hausdorff, we can find two disjoint open sets $O(ee') \ni ee'$ and $O'(e') \ni e'$. By the continuity of the semigroup operation on S, there are two neighborhoods $O(e)$ and $O(e') \subset O'(e')$ of the points e, e' in S such that $O(e) \cdot O(e') \subset O(ee')$. Since e is an accumulation point of the set E_C, we can find a number $n \in \omega$ such that $p^nq^n \in O(e)$. By a similar reason, there is a number $m \geq n$ such that $p^mq^m \in O(e')$. Then

$$O(e') \ni p^mq^m = p^nq^n \cdot p^mq^m \in O(e) \cdot O(e') \subset O(ee'),$$

which is not possible as $O'(e')$ and $O(ee')$ are disjoint.

Therefore the set E_C has a unique accumulation point e. We claim that the sequence $\{p^nq^n\}_{n=0}^\infty$ converges to the point e. Otherwise, we would find a neighborhood $O(e)$ such that the complement $E_C \setminus O(e)$ is infinite and hence has an accumulation point $e' \in S \setminus O(e)$, different from e, which is not possible.

This proves that the sequence $\{p^nq^n\}_{n=0}^\infty$ converges to e and hence the set $\bar{E}_C = E_C \cup \{e\}$ is compact and metrizable. Since the set $E = \{x \in S : xx = x\}$ of idempotents of S is closed, the accumulation point e of the set $E_C = E \cap \mathcal{C}(p,q)$ is an idempotent.

Next, we show that e commutes with all the elements of S. We start with the element p:

$$p \cdot e = p \cdot \lim_{k\to\infty} p^kq^k = \lim_{k\to\infty} p^{k+1}q^k = \lim_{k\to\infty} p^{k+1}q^{k+1}p = e \cdot p.$$

By analogy we can prove that $q\cdot e=e\cdot q$. Moreover,

$$pe\cdot eq=peq=p\cdot(\lim_{k\to\infty}p^kq^k)\cdot q=\lim_{k\to\infty}pp^kq^kq=\lim_{k\to\infty}p^{k+1}q^{k+1}=e,$$

which means that the elements $pe=ep$ and $qe=eq$ are mutually inverse elements of the maximal subgroup $H_e\subset S$ that contains the idempotent e. It follows that the element $a=pe$ generates a cyclic subgroup Z in S.

We claim that for every $n,m\in\omega$ we get $p^nq^m\cdot e=e\cdot p^nq^m=a^{n-m}$. Indeed, if $n\ge m$, then

$$\begin{aligned}p^nq^m\cdot e&=p^nq^m\cdot\lim_{k\to\infty}p^kq^k=\lim_{k\to\infty}p^nq^mp^kq^k=\lim_{k\to\infty}p^np^{k-m}q^k=\\&=\lim_{k\to\infty}p^{n-m}p^kq^k=p^{n-m}\lim_{k\to\infty}p^kq^k=p^{n-m}\cdot e=(pe)^{n-m}=a^{n-m}.\end{aligned}$$

On the other hand,

$$\begin{aligned}e\cdot p^nq^m&=\lim_{k\to\infty}p^kq^kp^nq^m=\lim_{k\to\infty}p^kq^{k-n}q^m=\lim_{k\to\infty}p^{n-m}p^{k-n+m}q^{k-n+m}=\\&=p^{n-m}\lim_{k\to\infty}p^{k-n+m}q^{k-n+m}=p^{n-m}\cdot e=(pe)^{n-m}=a^{n-m}.\end{aligned}$$

By analogy we can treat the case $n\le m$.

Therefore, e commutes with all the elements of the bicyclic semigroup $\mathcal{C}(p,q)$. Consequently, the closed subset $\{x\in S: xe=ex\}$ of S contains the dense subset $\mathcal{C}(p,q)$ of S and thus coincides with S, which means that the idempotent e commutes with all elements of S.

2. It follows that the map $\pi:S\to S$, $\pi:x\mapsto xe=ex$, is a continuous homomorphism. Let us show that $\pi(x)=x$ for every $x\in M=S\setminus\mathcal{C}(p,q)$. Assuming the converse, find $x\in M$ with $\pi(x)\ne x$. It is clear that $x\ne e$. Since S is Hausdorff, the points x,e and $\pi(x)=xe=ex$, have neighborhoods $O(x),O(e),O(\pi(x))\subset S$ such that $O(x)\cdot O(e)\cup O(e)\cdot O(x)\subset O(\pi(x))$ and $O(x)\cap O(\pi(x))=\emptyset$. Take any idempotent $p^kq^k\in O(e)\cap\mathcal{C}(p,q)$ and using the infinity of the intersection $O(x)\cap\mathcal{C}(p,q)$, pick a point $p^iq^j\in O(x)\cap\mathcal{C}(p,q)$ such that $i+j>2k$. Then either $i>k$ or $j>k$. If $i>k$, then

$$p^iq^j=p^kq^kp^iq^j\in O(x)\cap(O(e)\cdot O(x))\subset O(x)\cap O(\pi(x))=\emptyset.$$

If $j>k$, then

$$p^iq^j=p^iq^jp^kq^k\in O(x)\cap(O(x)\cdot O(e))\subset O(x)\cap O(\pi(x))=\emptyset.$$

Both cases lead to a contradition that completes the proof of the equality $\pi(x)=x$ for $x\in M$. This means that π retracts S onto $M=S\setminus\mathcal{C}(p,q)$. The latter set is a two-sided ideal in S according to Theorem 5.1(1).

3. As we have already proved, $\pi(p^nq^m)=a^{n-m}\in Z$ for every $n,m\in\omega$. Since $\mathcal{C}(p,q)$ is dense in S its image $Z=\pi(\mathcal{C}(p,q))$ is dense in $\pi(S)=M$.

4–5. The statements (4)–(5) have been proved in the first item.

6. Assume that S is a countably compact at $\mathcal{C}(p,q)$. By definition, S is countably pracompact. Next, we prove that the subgroup Z contains no non-trivial sequence that converges in S. Assume conversely that Z contains a non-trivial sequence $(a^{n_i})_{i=1}^\infty$ that converges to some point $x\in S$. We lose no generality assuming that the sequence (n_i) is strictly monotone and either consists of positive numbers and increases to $+\infty$ or else (n_i) consists of negative numbers and decreases to $-\infty$. The last case can be reduced to the first one by replacing the elements p, q by their places and then $a=\pi(p)$ will turn into $a^{-1}=\pi(q)$.

So, assume that the sequence (n_i) consists of positive numbers and is strictly increasing. First we prove that the sequence p^{n_i} converges to x. Assuming that this is not true, we could find a neighborhood $O(x)\subset S$ of x such that the complement $S\setminus O(x)$ contains an infinite subsequence $(p^{n_{i_k}})_{k\in\omega}$. This subsequence has an accumulation point $y\in M\setminus O(x)$ because S is countably compact at $\mathcal{C}(p,q)$. By the continuity of the homomorphism $\pi:S\to M$, we get

$$y=\pi(y)=\lim_{k\to\infty}\pi(p^{n_{i_k}})=\lim_{k\to\infty}a^{n_{i_k}}=x,$$

which is a contradiction. Thus the sequence (p^{n_i}) converges to x.

Since S is countably compact at $\mathcal{C}(p,q)$, the sequence $(q^{n_i})_{i\in\omega}$ has an accumulation point $y \in M$. Since $yx \in M \subset S \setminus \{p^0q^0\}$, there are neighborhoods $O(x), O(y) \subset S$ such that $O(y)\cdot O(x) \subset S \setminus \{1\}$. Since the sequence (p^{n_i}) converges to x, there is $i_0 \in \omega$ such that $p^{n_i} \in O(x)$ for all $i \ge i_0$. Since y is an accumulation point of the sequence $(q^{n_i})_{i\in\omega}$ there is a number $i \ge i_0$ such that $q^{n_i} \in O(y)$. The choice of i_0 guarantees that $p^{n_i} \in O(x)$. Then $1 = q^{n_i}p^{n_i} \in O(y)\cdot O(x) \subset S \setminus \{1\}$, which is a desired contradiction.

7. Finally, assume that S is countably compact at $\mathcal{C}(p,q)$ and M is Tychonov.

7a. It follows that the identity map $h : S \to \mathcal{C}(p,q) \cup_\pi M$ is bijective and continuous. Using the continuity of the homomorphism $\pi : S \to M$ and the discreteness of $\mathcal{C}(p,q)$ in S, it is easy to check that the space S is Tychonov.

Let $\beta h : \beta S \to \beta(\mathcal{C}(p,q)\cup_\pi M)$ be the Stone-Čech extension of h. Using the countable compactness of S at the dense subset $\mathcal{C}(p,q)$ of S, we can easily show that the spaces S and M are pseudocompact. By Proposition 1.5, the Stone-Čech compactification $\beta(\mathcal{C}(p,q)\cup_\pi M)$ can be identified with $\mathcal{C}(p,q)\cup_\pi \beta M$. It follows that βh is a homeomorphism and consequently, h is a homeomorphism.

7b. If the space M is openly factorizable, then so is the space $S = \mathcal{C}(p,q) \cup_\pi M$ according to Proposition 1.8. By Proposition 2.3 [3] the pseudocompactness and the open factorizability of S imply the open factorizability of βS. But thus contradicts Theorem 5.1(4).

7c. Assume that the square $M \times M$ is pseudocompact. Since the subgroup Z is dense in M, we can apply Theorem 2.5(2) to conclude that βM is a compact topological group. Compact topological groups, being Dugundji compact, are openly factorizable, which implies that βM is openly factorizable. By Propositions 2.3 of [3], the open factorizability of the Stone-Čech compactification βM implies the open factorizability of the space M. But this contradicts the preceding item.

7d. Assume that the (pseudocompact) semigroup M contains a dense totally bounded topological subgroup. By Theorem 2.4(1), the Stone-Čech compactification βM of M is a compact topological group. Further we continue as in the preceding item. □

6. A countably (pra)compact semigroup that contains $\mathcal{C}(p,q)$

In this section we shall construct a countably (pra)compact topological semigroup containing a bicyclic semigroup. Our main result is

Theorem 6.1. *The bicyclic subgroup is a subsemigroup of some Tychonov countably pracompact topological semigroup.*

The proof of this theorem relies on four lemmas.

Lemma 6.2. *A subgroup H of a topological group G contains a non-trivial convergent sequence if and only if H contains a non-trivial sequence that converges in G.*

Proof. Assume that some sequence $\{x_n\}_{n\in\omega} \subset H$ converges to a point $x_\infty \in G \setminus H$. By induction construct an increasing number sequence $(n_k)_{k\in\omega}$ such that

$$x_{n_{k+1}}x_{n_k}^{-1} \notin \{x_{n_{i+1}}x_{n_i}^{-1} : i < k\}$$

for every $k \in \omega$. Then $(x_{n_{k+1}}x_{n_k}^{-1})_{k\in\omega}$ is a non-trivial sequence in H that converges to the neutral element $e = x_\infty x_\infty^{-1}$. □

Lemma 6.3. *If a Tychonov space X is countably compact at an infinite subset $H \subset X$ that contains no non-trivial sequence that converges in X, then the closure $\mathrm{cl}_X(A)$ of any infinite subset $A \subset H$ has cardinality $|\mathrm{cl}_X(A)| \ge \mathfrak{c}$.*

Proof. We lose no generality assuming that the set A is countable. Since X is countably compact at A, the closure $\overline{A}$ of A in X contains a non-isolated point. We claim that $\overline{A}$ is not scattered. Assuming the opposite, we could find an isolated point a in the set $\bar{A}'$ of non-isolated points of $\overline{A}$. Choose a neighborhood $O(a) \subset X$ of a such that $O(a) \cap \bar{A}' = \{a\}$. It follows that a is a unique cluster point of

the set $A\cap O(a)$. Since X is countably compact at $H\supset A$ the set $A\cap O(a)$ converges to a in the sense that each neighborhood of a contains all but finite many points of $A\cap O(a)$. Choosing any bijective enumeration of the set $A\cap O(a)$ we obtain a non-trivial sequence in H that converges in X, but this is not possible.

Therefore, the set A is not scattered. Let K be any compactification of the space X and $\mathrm{cl}_K(A)$ be the closure of A in K. It follows that $\mathrm{cl}_K(A)$ is a non-scattered compact Hausdorff space. By [15, p.251] it admits a continuous surjective map $f:\mathrm{cl}_K(A)\to[0,1]$. Since A is dense in $\mathrm{cl}_K(A)$, we conclude that the set $f(A)$ is dense in $[0,1]=f(\mathrm{cl}_K(A))$. The countable compactness of $\overline{A}$ at A implies the countable compactness of $f(\overline{A})$ at the dense subset $f(A)$ of $[0,1]$. The latter is possible only if $f(\overline{A})$ coincides with $[0,1]$, which implies that $|\overline{A}|\ge|f(\overline{A})|\ge\mathfrak c$. $\square$

We define a subset L of an abelian group G to be *linearly independent* if the unique homomorphism $h:\mathrm{FA}(L)\to G$ from the free abelian group $\mathrm{FA}(L)$ generated by L into G such that $h|L=\mathrm{id}_L$ is one-to-one. For a linearly independent subset $L\subset G$ we shall identify the free abelian group $\mathrm{FA}(L)$ with the subgroup of G generated by L.

Lemma 6.4. *Let an Abelian torsion-free topological group G is countably compact at a subgroup $H\subset G$ that contains no non-trivial convergent sequence. Each linearly independent subset $L_0\subset G$ of size $|L_0|<\mathfrak c$ can be enlarged to a linearly independent subset $L\subset G$ of size $|L|=\mathfrak c$ such that the set $L\setminus L_0$ contains an accumulation point of each infinite subset $A\subset \mathrm{FA}(L)\cap H\subset G$.*

Proof. Without loss of generality, $L_0\ne\emptyset$. Take any faithfully indexed set $X=\{x_\alpha:\alpha<\mathfrak c\}$ of cardinality $|X|=\mathfrak c$ such that $X\cap L_0=\emptyset$ and consider the free abelian group $\mathrm{FA}(L_0\cup X)$ generated by the union $L_0\cup X$. For every ordinal $\alpha<\mathfrak c$ let $X_{<\alpha}=L_0\cup\{x_\beta:\beta<\alpha\}$ and $X_{\le\alpha}=L_0\cup\{x_\beta:\beta\le\alpha\}$. So, $L_0\cup X=X_{<\mathfrak c}$.

Denote by $\mathbf A$ the set of all countable subsets of the free abelian group $\mathrm{FA}(X_{<\mathfrak c})$. Since $|\mathrm{FA}(X_{<\mathfrak c})|=\mathfrak c$, the set $\mathbf A$ has size $|\mathbf A|=\mathfrak c^\omega=\mathfrak c$. To each countable subset $A\in\mathbf A$ assign the smallest ordinal $\xi(A)\le\mathfrak c$ such that $A\subset\mathrm{FA}(X_{<\xi(A)})$ and observe that $\xi(A)<\mathfrak c$ because $\mathfrak c$ has uncountable cofinality. It follows that $\xi(A)=0$ if and only if $A\subset\mathrm{FA}(L_0)$.

We claim that there is an enumeration $\mathbf A=\{A_\alpha:\alpha<\mathfrak c\}$ of the set $\mathbf A$ such that $\xi(A_\alpha)\le\alpha$ for every ordinal $\alpha<\mathfrak c$. To construct such an enumeration, first fix any enumeration $\mathbf A=\{A'_\alpha:\alpha<\mathfrak c\}$ such that $A'_0\subset\mathrm{FA}(L_0)$ and for every $A\in\mathbf A$ the set $\{\alpha<\mathfrak c:A'_\alpha=A\}$ has size continuum. Next, for every $\alpha<\mathfrak c$ put

$$A_\alpha=\begin{cases}A'_\alpha & \text{if } \xi(A_\alpha)\le\alpha\\ A'_0 & \text{otherwise.}\end{cases}$$

The identity inclusion $X_{<0}=L_0\subset G$ extends to a unique group homomorphism $h_{<0}:\mathrm{FA}(X_{<0})\to G$ which is injective because of the linear independence of L_0.

Inductively, for each ordinal $\alpha<\mathfrak c$ we shall construct an injective homomorphism $h_\alpha:\mathrm{FA}(X_{\le\alpha})\to G$ such that

- $h_\alpha|\mathrm{FA}(X_{\le\beta})=h_\beta$ for all $\beta<\alpha$;
- if $h_\alpha(A_\alpha)\subset H$, then the point $\bar x_\alpha=h_\alpha(x_\alpha)\in G$ is an accumulation point of the set $h_\alpha(A_\alpha)$.

We start with choosing a point $\bar x_\alpha$. Consider the injective group homomorphism $h_{<\alpha}:\mathrm{FA}(X_{<\alpha})\to G$ such that $h_{<\alpha}|\mathrm{FA}(X_{<\beta})=h_{<\beta}$ for $\beta<\alpha$. The image $h_{<\alpha}(\mathrm{FA}(X_{<\alpha}))$ is a free abelian subgroup of size $<\mathfrak c$ in G. Let $G_{<\alpha}=\big\{x\in G:\exists n\in\mathbb N\quad nx\in h_{<\alpha}(\mathrm{FA}(X_{<\alpha}))\big\}$ be the servant subgroup of the group $h_{<\alpha}(\mathrm{FA}(X_{<\alpha}))$ in G. Since G is torsion-free, $|G_{<\alpha}|\le\aleph_0\cdot|\mathrm{FA}(X_{<\alpha})|<\mathfrak c$.

Since the homomorphism $h_{<\alpha}:\mathrm{FA}(X_{<\alpha})\to G$ is injective, the set $B_\alpha=h_{<\alpha}(A_\alpha)$ is infinite. If $B_\alpha\subset H$, then by Lemmas 6.2 and 6.3, the closure $\overline{B}_\alpha$ of B_α in G has cardinality $|\overline{B}_\alpha|\ge\mathfrak c$. Consequently, we can find a point $\bar x_\alpha\in\overline{B}_\alpha\setminus G_{<\alpha}$. If $B_\alpha\not\subset H$, then take $\bar x_\alpha$ by any point of the set $\overline{H}\setminus G_{<\alpha}$. Such a point $\bar x_\alpha$ exists because the closure $\overline{H}$ of H in G has cardinality $|\overline{H}|\ge\mathfrak c>|G_{<\alpha}|$.

Such a choice of the point $\bar x_\alpha\notin G_{<\alpha}$ guarantees that the injective homomorphism $h_{<\alpha}$ extends to an injective homomorphism $h_\alpha:\mathrm{FA}(X_{\le\alpha})\to G$ such that $h_\alpha(x_\alpha)=\bar x_\alpha$. This completes the inductive step as well as the inductive construction.

The injectivity of the homomorphism $h_{<\mathfrak{c}}:\mathrm{FA}(X_{<\mathfrak{c}})\to G$ implies that the image $L=h_{<\mathfrak{c}}(X_{<\mathfrak{c}})$ of $X_{<\mathfrak{c}}=L_0\cup X$ is a linearly independent subset of G. By the choice of the homomorphism $h_{<0}$, we have $L_0=h_{<\mathfrak{c}}(L_0)\subset L$.

We claim that the subgroup $\mathrm{FA}(L)=h_{<\mathfrak{c}}(\mathrm{FA}(X_{<\mathfrak{c}})$ of G, generated by the set L, is countably compact at the subset $H\cap\mathrm{FA}(L)$. Take any countable infinite subset $B\subset H\cap\mathrm{FA}(L)$ and consider its preimage $A=h^{-1}_{<\mathfrak{c}}(B)\subset\mathrm{FA}(X_{<\mathfrak{c}})$. It follows that $A=A_\alpha$ for some $\alpha<\mathfrak{c}$. The choice of the point $\bar{x}_\alpha\in L\setminus L_0$ guarantees that $\bar{x}_\alpha$ is an accumulation point of the set $B=h_{<\mathfrak{c}}(A_\alpha)$. □

A (topological) semigroup S is called a (*topological*) *monoid* if S has a two-sided unit 1. The subgroup $H_1=\{x\in S:\exists y\in S\;\; xy=yx=1\}$ is called *the maximal subgroup* of a monoid S. For any subset B by $\mathrm{FM}(B)$ we denote the free abelian monoid generated by M. This is the subsemigroup of the free abelian group $\mathrm{FA}(B)$ generated by the set $B\cup\{1\}$, where 1 is the neutral element of $\mathrm{FA}(B)$.

Lemma 6.5. *Assume that a torsion-free Abelian topological group G is countably compact at a dense infinite cyclic subgroup $Z\subset G$ that contains no non-trivial convergent sequence. Then there is a Tychonov countably pracompact topological monoid M such that*

1. *M is algebraically isomorphic to the direct sum $\mathbb{Z}\oplus\mathrm{FM}(\mathfrak{c})$;*
2. *the maximal subgroup H_1 of M is cyclic, discrete, and dense in M;*
3. *$M\setminus H_1$ is an ideal in M;*
4. *M admits a continuous one-to-one homomorphism $h:M\to G$ such that $h(H_1)=Z$;*
5. *the semigroup M is countably compact if the group G is countably compact and contains no non-trivial convergent sequence.*

Proof. Let $H=Z$ if G is not countably compact and $H=G$ is G is countably compact. Let $a\in Z$ be a generator of the cyclic group Z. By Lemma 6.4, the linearly independent set $L_0=\{a\}$ can be enlarged to a linearly independent subset $L\subset G$ of size $|L|=\mathfrak{c}$ that generates the (free abelian) subgroup $\mathrm{FA}(L)$ in G such that for each infinite subset $A\subset H\cap\mathrm{FA}(L)\subset G$ the closure $\bar{A}$ meets the set $L\setminus L_0$. Let M be the subsemigroup of G generated by the set $\{-a,a\}\cup L$. Since each infinite subset of $Z\subset H\cap\mathrm{FA}(L)$ has an accumulation point (in $L\subset M$), the space M is countably compact at the subset Z. If G is countably compact, then $H=G$ and then M is countably compact because each infinite subset of $M\subset H\cap\mathrm{FA}(L)$ has an accumulation point in $L\subset M$.

It is clear that M is a monoid whose maximal subgroup H_1 coincides with Z and thus is dense in M. Also it is clear that M is algebraically isomorphic to $\mathbb{Z}\oplus\mathrm{FM}(\mathfrak{c})$.

Now we enlarge the topology τ on M induced from G in order to make the maximal subgroup $Z=H_1$ discrete. It is easy to see that the topology

$$\tau'=\{U\cup A:U\in\tau,\;A\subset Z\}$$

on M has the required property: Z becomes discrete but remains dense in this topology. It is easy to check that the space M endowed with this stronger topology remains a topological semigroup (this follows from the fact that $M\setminus Z$ is an ideal in M). Moreover, the topological space (M,τ') is Tychonov, see [10, 5.1.22].

It remains to check that the space (M,τ') is countably compact at H_1. Take any infinite subset $A\subset H_1=Z$. By Lemma 6.3, the closure $\bar{A}$ of A in the topology τ has size $|\bar{A}|\ge\mathfrak{c}$ and consequently, $\bar{A}$ contains a point $a\notin Z$. It follows from the definition of the topology τ' that the point a remains an accumulation point of the set A in the topology τ'.

If the group G is countably compact, then so is the semigroup M and the preceding argument ensures that M remains countably compact in the stronger topology τ'. □

Now we are ready to present the
Proof of Theorem 6.1. Fix an Abelian torsion-free topological group G which is countably compact at a dense infinite cyclic subgroup $Z\subset G$ containing no non-trivial convergent sequence. For G we can take the Bohr compactification $b\mathbb{Z}$ of the group of integers $\mathbb{Z}$ and for Z the image $\mathbb{Z}^\sharp$ of $\mathbb{Z}$ in

$b\mathbb{Z}$. It is well-known that the Bohr compactification $b\mathbb{Z}$ is torsion-free and its subgroup $\mathbb{Z}^\sharp$ contans no non-trivial convergent sequence, see [8] or [11].

By Lemmas 6.4 and 6.5, there is a commutative Tychonov countably pracompact topological monoid M whose maximal subgroup H_1 is cyclic, discrete, and dense in M. In addition, $M \setminus H_1$ is an ideal in M. Let $h : \mathbb{Z} \to H_1$ be any isomorphism. Define a homomorphism $\pi : \mathcal{C}(p, q) \to M$ letting $\pi(p^n q^m) = h(n-m)$ for $n, m \in \omega$. By Theorem 4.1(3) the semitopological semigroup $S = \mathcal{C}(p, q) \cup_\pi M$ is a topological semigroup. By Propositions 1.1 and 1.6, the space S is Tychonov and countably pracompact.

Moreover, if the group G is countably compact and contains no non-trivial convergent sequence, then the semigroup M is countably compact according to Lemma 6.5(5), and then the semigroup S is countably compact by Proposition 1.4. □

Let us remark that the above proof yields a bit more than required in Theorem 6.1, namely:

Theorem 6.6. *If there is a torsion-free Abelian countably compact topological group G without non-trivial convergent sequences, then there exists a Tychonov countably compact semigroup S containing a bicyclic semigroup.*

The first example of a group G with properties required in Theorem 6.6 was constructed by M. Tkachenko under the Continuum Hypothesis [26]. Later, the Continuum Hypothesis was weakened to the Martin Axiom for σ-centered posets by A. Tomita in [27], for countable posets in [19], and finally to the existence continuum many incomparable selective ultrafilters in [20]. Yet, the problem of the existence of a countably compact group without convergent sequences in ZFC seems to be open, see [7].

Those consistency results combined with Theorem 6.6 imply

Corollary 6.7. *The Martin Axiom implies the existence of a Tychonov countably compact topological semigroup S that contains a bicyclic semigroup.*

Remark 6.8. By Theorem 5.1(5), the almost periodic compactification $\mathrm{AP}(S)$ of the countably (pra)compact semigroup $S \supset \mathcal{C}(p, q)$ constructed in Theorem 6.6 (or 6.1) is a compact topological group. Consequently, the canonic homomorphism $\eta : S \to \mathrm{AP}(S)$ is not injective in contrast to the canonic homomorphism $\eta : S \to \mathrm{WAP}(S) = \beta S$ which is a topological embedding by Theorem 2.2. In particular, S is a countably (pra)compact topological semigroup that does not embed into a compact topological semigroup.

7. Some Open Problems

The consistency nature of Theorem 6.6 and Corollary 6.7 suggests:

Problem 7.1. *Is there a ZFC-example of a countably compact topological semigroup that contains the bicyclic semigroup?*

Theorem 6.1 gives an example of a countably compact topological semigroup S for which the canonical homomorphism $\eta : S \to \mathrm{AP}(S)$ is not injective.

Problem 7.2. *Is there a non-trivial countably (pra)compact topological semigroup S whose almost periodic compactification $\mathrm{AP}(S)$ is a singleton?*

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References

[1] A.V. Arkhangel'skiĭ, *Topological function spaces*, Kluwer Publ., Dordrecht, 1992.

[2] O. Andersen, *Ein Bericht über die Struktur abstrakter Halbgruppen*, PhD Thesis, Hamburg, 1952.

[3] T. Banakh, S.D. Dimitrova, *Openly factorizable spaces and compact extensions of topological semigroups*, preprint (arXiv:0811.4272).

[4] T.Banakh, S.Dimitrova, O.Gutik, *The Rees-Suschkewitsch Theorem for simple topological semigroups*, Mat. Stud. **31**:2 (2009) 211–218.

[5] J. Berglund, H. Junghenn, P. Milnes, *Analysis on semigroups. Function spaces, compactifications, representations*, A Wiley-Intersci. Publ. John Wiley & Sons, Inc., New York, 1989.

[6] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Vol. I. Amer. Math. Soc. Surveys 7, 1961; Vol. II. Amer. Math. Soc. Surveys 7, 1967.

[7] D. Dikranjan, D. Shakhmatov, *Selected topics in the structure theory of topological groups*, in: Open Problems in Topology, II (E. Pearl ed.), Elsevier, 2007. p.389–406.

[8] E. van Douwen, *The maximal totally bounded group topology on G and the biggest minimal G-space, for abelian groups G*, Topology Appl. **34**:1 (1990), 69–91.

[9] C. Eberhart, J. Selden, *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc. **144** (1969), 115–126.

[10] R. Engelking, *General Topology*, 2nd ed., Heldermann, Berlin, 1989.

[11] J. Galindo, S. Hernández, Ta-Sun Wu, *Recent results and open questions relating Chu duality and Bohr compactifications of locally compact groups*, in: Open Problems in Topology, II (E. Pearl ed.), Elsevier, 2007. p.407–422.

[12] S. Garcia-Ferreira, A. H. Tomita, S. Watson, *Countably compact groups from a selective ultrafilter*, Proc. Amer. Math. Soc. **133** (2005), 937–943.

[13] O. Gutik, D. Pagon, D. Repovš, *On sequential countably compact topological semigroups*, preprint (http://arxiv.org/abs/0804.1449v1)

[14] O.Gutik, D.Repovs, *On countably compact 0-simple topological inverse semigroups*, Semigroup Forum **75**:2 (2007), 464–469.

[15] P. Habala, P. Hájek, V. Zizler, *Introduction to Banach spaces*, Praha, Matfyzpress, 1996.

[16] J.A. Hildenbrandt, R.J. Koch, *Swelling actions of Γ-compact semigroups*, Semigroup Forum **33**:1 (1988), 65–85.

[17] N. Hindman, D. Strauss, *Algebra in the Stone-Čech compactification. Theory and applications,* de Gruyter Expositions in Mathematics, 27. Walter de Gruyter & Co., Berlin, 1998.

[18] R.J. Koch, A.D. Wallace, *Stability in semigroups*, Duke Math. J. **24** (1957), 193–195.

[19] P. Koszmider, A. Tomita, S. Watson, *Forcing countably compact group topologies on a larger free Abelian group*, Topology Proc. **25** (2000), 563–574.

[20] R. Madariaga-Garcia, A.H. Tomita, *Countably compact topological group topologies on free Abelian groups from selective ultrafilters*, Topology Appl. **154** (2007), 1470–1480.

[21] M. Megrelishvili, *Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial*, Semigroup Forum **63**:3 (2001), 357–370.

[22] D. Rees, *On semi-groups*, Proc. Cambridge Phil. Soc. **36** (1940), 387–400.

[23] E.A. Reznichenko, *Extension of functions defined on products of pseudocompact spaces and continuity of the inverse in pseudocompact groups*, Topology Appl. **59**:3 (1994), 233–244.

[24] D. Robbie, S. Svetlichnyi, *An answer to A. D. Wallace's question about countably compact cancellative semigroups*, Proc. Amer. Math. Soc. **124**:1 (1996), 325–330.

[25] W. Ruppert, *Compact Semitopological Semigroups: An Intrinsic Theory*, LNM **1079**, Springer, 1984.

[26] M. Tkachenko, *Countably compact and pseudocompact topologies on free Abelian groups*, Soviet Math. (Iz. VUZ) **34**:5 (1990), 79–86.

[27] A.H. Tomita, *The Wallace problem: A counterexample from $MA_{\text{countable}}$ and p-compactness*, Canad. Math. Bull. **39**:4 (1996), 486–498.

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